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Self-similar behavior of plasma fluid equations*

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SUMMARY

Using the "s-parameter groups of transformations" technique the self-similar behavior for three sets of commonly used plasma fluid model equations is ascertained. The self-similar variables are $\xi = x/t^A$, $\xi = t/x^B$ or $\xi = t\exp(-\alpha x)$ where A, B and α are numbers and x and t are space and time variables. The resulting ordinary differential equations are obtained. By judicious choices of the parameters, a partial integration of the equations is obtained, thus displaying the analytic character of the systems.

1. Introduction

It is well known that self-similar (invariant) analysis of nonlinear partial differential equations sometimes leads to rigorous solutions for problems in fluid mechanics, diffusion and wave propagation. The details of the general theory, many references and a variety of examples can be found in the literature (Ames [3]). They sometimes lead to useful scaling laws (Moran [12]) or asymptotic states (Serrin [16]; Peletier [15]) of non-similar problems. The importance of self-similar solutions has been exemplified in physical systems governed by parabolic and hyperbolic equations (Ovsjannikov [14]; Hansen [8]; Ames [3]). The mathematical implications of self-similarity is the existence of a transformation(s) of variables which achieves a reduction in the number of independent variables in a system of equations. For example a partial differential equation in spatial and temporal independent variables x and t may be transformed to an ordinary differential equation in ξ where $\xi = \xi(x, t)$. The purpose of this work is to demonstrate how the form of ξ and functional dependence upon ξ is found for a fluid model description of a plasma which includes Poisson's equation. Herein an assumed group will be employed, thus establishing the general, but not specific, form of ξ. In a later paper we will address the problem of determining all permissible groups.

A few examples of self-similarity have appeared in the plasma literature (Gurevich *et al.* [6]; Friedhoffer [5]; Korn *et al.* [10]; Allen and Andrews [2]; Alexeff *et al.* [1]; Anderson *et al.* [4]). In these works, with the notable exception of Friedhoffer, the self-similar variable has almost invariably been asserted to be $\xi = x/t$. In an earlier paper (Ikezi *et al.* [9]), we have shown that an ion acoustic wave, which could be modeled by a Korteweg–deVries equation, permits a self-similar variable,

$$\xi = \frac{x - V_A t}{(x/\lambda_D)^{\frac{1}{3}}}\left(\frac{2}{3}\right)^{\frac{1}{3}},$$

where V_A and λ_D are the ion acoustic velocity and Debye length, respectively.

Section 2 presents the three sets of equations which will be examined to ascertain their self-similar behavior. The sets of equations are: (a) multiple species fluid equations truncated at the third moment plus Poisson's equation; (b) massless isothermal electron fluid and cold nonlinear ion fluid plus Poisson's equation; and for completeness, (c) massless isothermal electron fluid and cold nonlinear ion fluid with a quasi-neutrality assumption.

In Section 3, we describe the procedure for finding the self-similar variables for these three

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sets of equations using the technique of "s-parameter groups of transformations" (Ames [3]). There it is shown that $\xi = x/t^A$ or the equivalent $\xi = t/x^B$ or $\xi = t\exp(-\alpha x)$, where A, B and α are numbers that can be adjusted to satisfy boundary or initial conditions, some physical constraint (Ames [3]) or for integration of the complete system (Lonngren *et al.* [11]).

Section 4 contains a discussion of some solutions of the similar equations and Section 5 presents the summary and conclusions.

2. Equations studied

Herein we restrict attention to a fluid description of an unmagnetized plasma in an electrostatic approximation. The first set of equations to be studied is that of a hot plasma represented by the multiple species fluid equations plus Poisson's equation. When the fluid equations are truncated at the third moment, these become

$$\frac{\partial n_l}{\partial t} + \frac{\partial}{\partial x}(n_l v_l) = 0\,, \tag{1.1}$$

$$\frac{\partial v_l}{\partial t} + v_l \frac{\partial v_l}{\partial x} + \frac{1}{m_l}\frac{1}{n_l}\frac{\partial P_l}{\partial x} = \frac{q_l}{m_l} E\,, \tag{1.2}$$

$$\frac{\partial P_l}{\partial t} + \gamma_l P_l \frac{\partial v_l}{\partial x} + v_l \frac{\partial P_l}{\partial x} = 0\,, \tag{1.3}$$

$$\frac{\partial E}{\partial x} = \sum_l \frac{q_l}{\varepsilon_0} n_l\,. \tag{1.4}$$

All the symbols employed in (1) are standard and the subscript l designates the lth species of charged particles in the plasma and γ_l denotes a compression constant which may be different for the different species. This is a set of equations which has been widely used in the study of plasma waves, plasma shocks, etc.

The second set of equations can be obtained from (1) by assuming the electrons to be a linear massless isothermal fluid and the ions to be a cold nonlinear fluid. The electrons can be described by the equation

$$E = -\frac{KT_e}{n_0 e}\frac{\partial}{\partial x} n_e\,,$$

where n_0 is the average electron density. The cold ions can be described by the first two moment equations. Thus we write the closed set of equations as

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i v_i) = 0\,, \tag{2.1}$$

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} + \frac{V_a^2}{n_0}\frac{\partial n_e}{\partial x} = 0\,, \tag{2.2}$$

$$\lambda_D^2 \frac{\partial^2 n_e}{\partial x^2} = n_e - n_i\,, \tag{2.3}$$

where $V_a = (KT_e/m_i)^{\frac{1}{2}}$ is the ion acoustic speed and $\lambda_D = (\varepsilon_0 KT_e/n_0 e^2)^{\frac{1}{2}}$ is the electron Debye length. Equations (2) are widely employed in the study of ion acoustic waves. This system contains the simple nonlinear effects from the terms $\partial(n_i v_i)/\partial x$ and $v_i \partial v_i/\partial x$ as well as the lowest order dispersive effects as represented by the term $\lambda_D^2 \partial^2 n_e/\partial x^2$. Studies of (2), in the small perturbation region, result in a Korteweg–deVries equation (Washimi and Taniuti [17]).

The third set of equations,

$$\frac{\partial n}{\partial t} + \frac{\partial(nv)}{\partial x} = 0\,, \tag{3.1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{V_A^2}{n_0}\frac{\partial n}{\partial x} = 0\,, \tag{3.2}$$

is similar to the second except Poisson's equation is neglected, that is quasi-neutrality is assumed. This set, which is valid for long wavelength phenomena ($\lambda \gg \lambda_D$), is the set that has received some attention in the literature for possible self-similar behavior. We include it here for completeness although we note that the quasi-neutrality assumption is not valid as soon as the nonlinearity becomes sufficiently large to cause effects such as steepening of a wave-front where short wavelengths effects are bound to occur (Gurevich *et al.* [6], [7]).

3. Self-similarity via one-parameter transformation groups

There exists a simple effective procedure for ascertaining (if any exist) similarity variables for a set of partial differential equations. Growing from the Lie theory of groups it is known that these similarity variables are identical to the invariants of a particular one (or more) parameter group of transformations (details and references found in Ames [3]). Since this is not a standard method in plasma physics, we briefly outline a simplified procedure, using the linear and spiral groups, in this section. In a later paper the determination of possible groups leaving certain sets of plasma equations invariant will be presented.

In particular a set of continuous transformations (expansions or contractions) with a positive real parameter, a,

$$G:\begin{cases} \bar{x} = a^{\alpha_1}x\,, \quad \bar{t} = a^{\alpha_2}t \\ \bar{n}_l = a^{\beta_1}n_l\,, \quad \bar{v}_l = a^{\beta_2}v_l\,, \quad \bar{P}_l = a^{\beta_3}P_l\,, \quad \bar{E} = a^{\beta_4}E \end{cases} \tag{4}$$

form a group (the "linear" group) under the composition operation. Here the α_i's and β_j's are to be determined so that the set of equations is "(absolutely) constant conformally invariant" under the group G (Ames [3]). A function $F_j(x_i)$ is said to "constant conformally invariant" (CCI) under G if $F_j(x_i) = f_j(a)F_j(\bar{x}_i)$, where f_j is some function of the parameter a. If $f_j(a) \equiv 1$ the constant conformal invariance is called "absolute". The requirement that equation (1) be CCI under G is satisfied if

$$\beta_1 = -2\alpha_2\,, \quad \beta_2 = \alpha_1 - \alpha_2\,, \quad \beta_3 = 2\alpha_1 - 4\alpha_2\,, \quad \beta_4 = \alpha_1 - \alpha_2\,. \tag{5}$$

The result from transforming equation (1) into the variables, $\bar{x}$, $\bar{t}$, $\bar{n}_l$, $\bar{v}_l$, $\bar{P}_l$ and $\bar{E}_l$. For example (1.2) gives

$$a^{-\beta_2+\alpha_2}\frac{\partial \bar{v}_l}{\partial \bar{t}} + a^{-2\beta_2+\alpha_1}\bar{v}_l\frac{\partial \bar{v}_l}{\partial \bar{x}} + a^{-\beta_3+\beta_1+\alpha_1}\frac{1}{m_l}\frac{\partial \bar{P}_l}{\partial \bar{x}}\frac{1}{\bar{n}_l} = a^{-\beta_4}\frac{e_l}{m_l}\bar{E} \tag{6}$$

The condition that (6) is constant conformally invariant under the transformation group G means

$$-\beta_2 + \alpha_2 = -2\beta_2 + \alpha_1 = -\beta_3 + \beta_1 + \alpha_1 = -\beta_4\,. \tag{7}$$

For each of the equations in (1), an equation analogous to (7) is found. After some simple algebra, (5) is obtained.

Second, we determine the invariants of the transformation group G. This is achieved by employing a well known theorem from group theory (Ames [3]). The invariants are obtained from $QI \equiv 0$ where I is an invariant and Q is the operator

$$\begin{aligned} Q \equiv {} & \left.\frac{\partial \bar{x}}{\partial a}\right|_{a=1}\frac{\partial}{\partial x} + \left.\frac{\partial \bar{t}}{\partial a}\right|_{a=1}\frac{\partial}{\partial t} + \left.\frac{\partial \bar{n}_l}{\partial a}\right|_{a=1}\frac{\partial}{\partial n_l} + \left.\frac{\partial \bar{v}_l}{\partial a}\right|_{a=1}\frac{\partial}{\partial v_l} + \left.\frac{\partial \bar{P}_l}{\partial a}\right|_{a=1}\frac{\partial}{\partial P_l} + \left.\frac{\partial \bar{E}_l}{\partial a}\right|_{a=1}\frac{\partial}{\partial E_l} \\ = {} & \alpha_1 x\frac{\partial}{\partial x} + \alpha_2 t\frac{\partial}{\partial t} + \beta_1 n_l\frac{\partial}{\partial n_l} + \beta_2 v_l\frac{\partial}{\partial v_l} + \beta_3 P_l\frac{\partial}{\partial P_l} + \beta_4 E\frac{\partial}{\partial E}\,. \end{aligned} \tag{8}$$

Solutions of $QI = 0$ are obtained by solving the Lagrange subsidiary equations

$$\frac{dx}{\alpha_1 x} = \frac{dt}{\alpha_2 t} = \frac{dn_l}{\beta_1 n_l} = \frac{dv_l}{\beta_2 v_l} = \frac{dP_l}{\beta_3 P_l} = \frac{dE}{\beta_4 E} = \frac{dI}{0}. \tag{9}$$

Equation (9) provides all of the group invariants (five in this case) which must employ the restrictions given in (5). Thus we obtain the invariants of the group G subject to the condition that (1) is constant conformally invariant under G and according to the theorem developed by Morgan [13]. These invariants of the group G are also the self-similar variables for the original partial differential equations. For (1) these are found to be

$$\xi = \frac{x}{t^A}, \quad N_l = n_l t^2, \quad U_l = v_l t^{1-A},$$
$$\mathscr{P}_l = P_l t^{4-2A}, \quad \varepsilon = E t^{2-A}, \tag{10}$$

where N_l, U_l, $\mathscr{P}_l$, ε are only functions of ξ, and $A = \alpha_1/\alpha_2$ is an arbitrary parameter, the selection of which may be subject to physical restrictions and/or the boundary and the initial conditions*. It is a simple matter to check now that equation (10) does indeed give the self-similar variables for (1) by direct substitution. Thus, we obtain

$$\frac{d}{d\xi}(N_l U_l) - A\xi \frac{d}{d\xi} N_l - 2N_l = 0, \tag{11.1}$$

$$-A\xi \frac{d}{d\xi} U_l + (A-1) U_l + U_l \frac{d}{d\xi} U_l + \frac{1}{m_l N_l} \frac{d}{d\xi} \mathscr{P}_l + \frac{e_l}{m_l} \varepsilon = 0, \tag{11.2}$$

$$(2A-4)\mathscr{P}_l - A\xi \frac{d}{d\xi} \mathscr{P}_l + \gamma_l \mathscr{P}_l \frac{d}{d\xi} U_l + U_l \frac{d}{d\xi} \mathscr{P}_l = 0, \tag{11.3}$$

$$\frac{d}{d\xi} \varepsilon = \sum_l \frac{e_l}{\varepsilon_0} N_l. \tag{11.4}$$

If we try the above procedure on (2), we find that $\alpha_1 = 0$. Therefore the group G is not suitable for equation (2). However, we can select another transformation group G_0 (the "spiral" group) with a nonzero real parameter a, whose elements are

$$G_0: \begin{cases} \bar{x} = x + \log_e a, & \bar{n}_i = a^{\beta_1} n_i, \\ \bar{t} = a^{\alpha} t, & \bar{n}_e = a^{\beta_1} n_e, \quad \bar{v}_i = a^{\beta_2} v_i. \end{cases} \tag{12}$$

The requirement that (2) be constant conformally invariant under the transformation group G_0 is that

$$\beta_1 = -2\alpha, \qquad \beta_2 = -\alpha. \tag{13}$$

The invariants of the group G_0 are obtained in a manner analogous to that employed for G. If we require $QI = 0$, then we obtain

$$\frac{dx}{1} = \frac{dt}{\alpha t} = \frac{dn_i}{\beta_1 n_i} = \frac{dn_e}{\beta_1 n_e} = \frac{dv_i}{\beta_2 v_i} = \frac{dI}{0}. \tag{14}$$

Thus the similar variables for (2) are

$$\xi = t\exp(-\alpha x), \quad N_i = n_i \exp(2\alpha x),$$
$$N_e = n_e \exp(2\alpha x), \quad U = v_i \exp(\alpha x), \tag{15}$$

where α is an arbitrary parameter and N_i, N_e, and U are only functions of ξ.

Substituting (15) into (2), we obtain the set of ordinary differential equations

$$\left(\frac{1}{\alpha} - \xi U\right) \frac{dN_i}{d\xi} - \xi N_i \frac{dU}{d\xi} - 3N_i U = 0, \tag{16.1}$$

$$\left(\frac{1}{\alpha} - \xi U\right) \frac{dU}{d\xi} - \xi \frac{dN_e}{d\xi} - 2N_e - U^2 = 0, \tag{16.2}$$

* The self-similar variable $\xi = t/x^B$ where $B = \alpha_2/\alpha_1$ follows directly from (9) and will not be treated.

$$\xi^2 \frac{d^2 N_e}{d\xi^2} + 5\xi \frac{dN_e}{d\xi} + \left(4 - \frac{1}{\alpha^2}\right) N_e + \frac{1}{\alpha^2} N_i = 0 . \tag{16.3}$$

To examine the third set (3), we again use the group G,

$$\bar{x} = a^{\alpha_1} x , \quad \bar{n} = a^{\beta_1} n ,$$
$$\bar{t} = a^{\alpha_2} t , \quad \bar{v} = a^{\beta_2} v , \tag{17}$$

and find using the above procedure that

$$\xi = \frac{x}{t^A}, \quad U = \frac{v}{t^{A-1}}, \quad N = \frac{n}{t^{2(A-1)}}, \tag{18}$$

where U and N satisfy the ordinary differential equations

$$-A\xi \frac{dN}{d\xi} + 2(A-1)N + \frac{d(NU)}{d\xi} = 0 , \tag{19.1}$$

$$(A-1)U - A\xi \frac{dU}{d\xi} + U \frac{dU}{d\xi} + \frac{V_A^2}{n_0} \frac{dN}{d\xi} = 0 , \tag{19.2}$$

and A is an arbitrary parameter. With $A=1$, this reduces to the set of equations studied previously (Gurevich *et al.* [6]; Korn *et al.* [10]; Allen and Andrews [2]; Alexeff *et al.* [1]; Anderson *et al.* [4]).

Equations (10), (15), and (18) are a set of self-similar variables for the three sets of fluid plasma equations. Since equations (1)–(3) are invariant with respect to translational transformation in both space and time, all of the self-similar variables can be appropriately modified by replacing x and t by $x+x_0$ and $t+t_0$ where x_0 and t_0 are constants which are adjustable according to the problem at hand. A general rule to be borne in mind is that we would like to keep and create as many free parameters as possible in order to satisfy any given initial or boundary conditions or to allow one to integrate the equations as discussed in the next section.

4. Comments on self-similar equations

Certain choices of the free parameter (A) may permit partial or complete integration of the self-similar equations. We consider several possibilities.

Examining equation (11), it is clear that the choice $A=2$ permits (11.1) to be integrated to (we have dropped the subscripts and assumed a one species plasma for simplicity in illustration)

$$N = \frac{c_1}{U-2\xi}, \tag{20.1}$$

whereupon (11.3) becomes the separable form

$$\gamma \mathscr{P} \frac{dU}{d\xi} + (U-2\xi) \frac{d\mathscr{P}}{d\xi} = 0 .$$

This integrates to

$$\mathscr{P} = c_2 e^{-\gamma\xi} \; ; \; \text{i.e., } P = c_2 e^{-\gamma x/t^2} \tag{20.2}$$

and

$$U = c_3 e^{\xi} + 2\xi + 2 \; ; \; \text{i.e., } v = t[c_3 e^{x/t^2} + 2(x/t^2) + 2] , \tag{20.3}$$

whereupon

$$N = \frac{c_1}{c_3 e^{\xi} + 2} \; ; \; \text{i.e., } n = t^{-2} c_1/(c_3 e^{x/t^2} + 2) ,$$

and the c_j are arbitrary constants.

Lastly we use (11.2) to calculate the field ε and find

$$\varepsilon = \frac{m}{e}\left[2+10\xi+4\xi^2+c_3(4\xi+1)e^{\xi}+\frac{\gamma c_2}{mc_1}(c_3 e^{\xi}+2)e^{-\gamma\xi}\right]. \tag{20.4}$$

Unfortunately, Poisson's equation (11.4) is not, however, identically satisfied for this choice of A.

Examining now equation (19) it is seen that (19.1) is integrable immediately if $A=\frac{2}{3}$ for then we find

$$U = \frac{c_1}{N} + \tfrac{2}{3}\xi \tag{21.1}$$

and (19.2) becomes

$$9(\lambda N^3 - c_1^2)\frac{dN}{d\xi} + 3c_1 N^2 - 2\xi N^3 = 0. \tag{21.2}$$

Alternatively, the choice of $A=\frac{1}{2}$ leads to the immediate integration of (19.2) to

$$-\tfrac{1}{2}\xi U + \tfrac{1}{2}U^2 + \lambda N = c_2, \tag{22.1}$$

whereupon (19.1) becomes

$$(3U^2 - 3\xi U + \tfrac{1}{2}\xi^2 - 2c_1)\frac{dU}{d\xi} + (2c_1 + \tfrac{3}{2}\xi U - 2U^2) = 0. \tag{22.2}$$

In both cases at least one quadrature (but not two) appears to be necessary.

5. Conclusions

In this paper, we have examined the self-similar nature of three sets of fluid equations which have found considerable application in plasma physics. In these three cases, we found several free constants which could be specified by some boundary or initial condition or some other physical constraint. On occasion, a judicious choice may lead to a simplification or even a total integration of the system. There does not yet appear to be a methodical way in which to incorporate them at the start of a calculation.

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